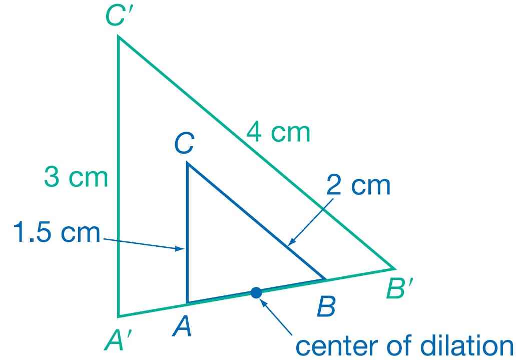
LESSON 10

Dilations and Similarity

Defining Dilation and Similarity

UNDERSTAND A **dilation** is a transformation that moves the points of a line, line segment, or figure either toward or away from a point called the **center of dilation**. The center of dilation can be any point inside the figure, on the figure, or outside the figure.

In the diagram on the right, blue triangle *ABC* was dilated to produce green triangle $A'B'C'$. The figures have the same shape, but $\triangle A'B'C'$ is twice the size of $\triangle ABC$.

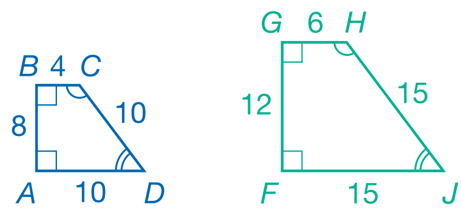


If the center of dilation lies on a line or line segment, the dilated image of the line will be collinear with its preimage. So $\overline{A'B'}$ is collinear with its corresponding side, $\overline{AB}$. If a line or line segment does not pass through the center of dilation, the dilated image will be parallel to the preimage. So, $\overline{B'C'} \parallel \overline{BC}$ and $\overline{A'C'} \parallel \overline{AC}$. (The symbol $\parallel$ means "is parallel to.")

When a dilation is applied to a line segment or closed figure, it changes the size of the image according to a **scale factor**, k. If $k > 1$, the figure is enlarged. If $0 < k < 1$, the figure is reduced. In the figure above, $\triangle ABC$ was dilated by a scale factor of 2, so it was enlarged. Each side of $\triangle A'B'C'$ is twice as long as its corresponding side on $\triangle ABC$.

UNDERSTAND Dilating a figure produces a figure that is the same shape as the original figure, but a different size. Like rigid motions, dilations preserve angle measures. Unlike rigid motions, dilations do not preserve the lengths of line segments. Instead, they produce a figure with sides that are proportional to the sides of the preimage. So, the original figure and its dilated image are **similar** figures.

Trapezoid *ABCD* was dilated by a scale factor of $\frac{3}{2}$ to form trapezoid *FGHJ* on the right. The angle marks show that corresponding angles are congruent. Corresponding side lengths are proportional. So, $ABCD \sim FGHJ$. (The symbol $\sim$ means "is similar to.")



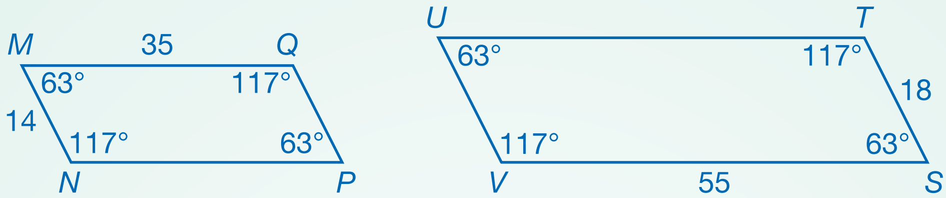
$\angle F \cong \angle A \quad \angle G \cong \angle B \quad \angle H \cong \angle C \quad \angle J \cong \angle D$

$\frac{FG}{AB} = \frac{12}{8} = \frac{3}{2} \quad \frac{GH}{BC} = \frac{6}{4} = \frac{3}{2} \quad \frac{HJ}{CD} = \frac{15}{10} = \frac{3}{2} \quad \frac{FJ}{AD} = \frac{15}{10} = \frac{3}{2}$

A **regular polygon** is a polygon in which all sides have the same length and all angles have the same measure. Any two regular polygons of the same type—having the same number of sides—are similar to each other.

Connect

Is parallelogram *MNPQ* ~ parallelogram *STUV*? Explain how you know.



1 Compare corresponding angle measures.

Recall that the order of the letters in the names of the figures identifies corresponding vertices. So, vertex *M* corresponds to vertex *S*, vertex *N* corresponds to vertex *T*, and so on.

$m\angle M = m\angle S = 63°$

$m\angle N = m\angle T = 117°$

$m\angle P = m\angle U = 63°$

$m\angle Q = m\angle V = 117°$

Corresponding angles are equal in measure, or congruent.

2 Determine if corresponding sides have proportional lengths.

Opposite sides of parallelograms are congruent.

$MQ = NP = 35$ $TU = SV = 55$

$MN = PQ = 14$ $ST = UV = 18$

Compare corresponding sides to determine if they are proportional.

$\frac{MQ}{SV} = \frac{35}{55} = \frac{7}{11}$

$\frac{MN}{ST} = \frac{14}{18} = \frac{7}{9}$

$\frac{NP}{TU} = \frac{MQ}{SV} = \frac{35}{55} = \frac{7}{11}$

$\frac{PQ}{UV} = \frac{MN}{ST} = \frac{14}{18} = \frac{7}{9}$

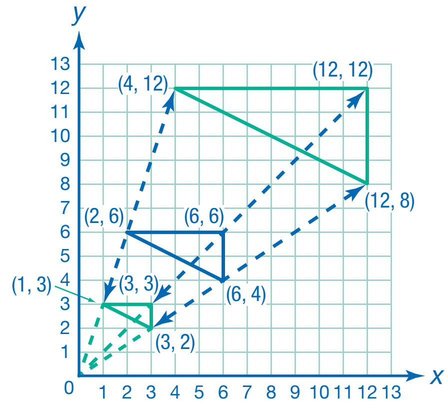
▶ Not all corresponding sides are in the same ratio, so parallelogram *MNPQ* is not similar to parallelogram *STUV*.

DISCUSS

How could you change parallelogram *STUV* to make it similar to parallelogram *MNPQ*? Is that the only way to make the two parallelograms similar?

Dilations as Functions

UNDERSTAND The scale factor describes how the length of a line segment changes during a dilation. It also describes how the distance from the center of dilation to a given point changes during that dilation. The graph on the right shows the blue triangle dilated by a scale factor of 0.5 and a scale factor of 2 with the center of dilation at the origin. Notice that each vertex of the larger green triangle is twice as far from the origin as the corresponding vertex on the blue triangle, and each vertex of the smaller green triangle is half as far from the origin.



A dilation on the coordinate plane can be written as a function. The input of this function is a point on the coordinate plane, (x, y). When you apply the function to a point, the output of the function will be the coordinates of the dilated image of that point.

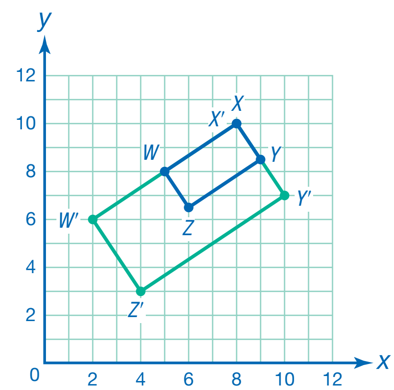
To dilate a point on a figure (x, y) by a scale factor k with the center of dilation at (a, b), use the following rule:

$$D_k(x, y) = (a + k(x - a), b + k(y - b))$$

When the center of dilation is the origin, $a = 0$ and $b = 0$, so this rule simplifies to:

$$D_k(x, y) = (kx, ky)$$

The graph to the right shows a dilation of rectangle $WXYZ$ with the center of dilation at $X(8, 10)$ and a scale factor of 2. The function that represents this transformation is

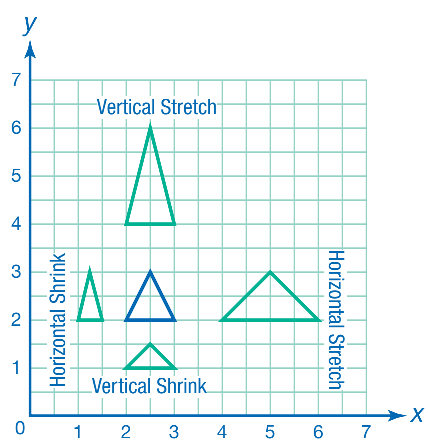


$$D_2(x, y) = (8 + 2(x - 8), 10 + 2(y - 10)) = (8 + 2x - 16, 10 + 2y - 20) = (2x - 8, 2y - 10)$$

Point W has coordinates (5, 8). Substitute these coordinates into the function to find W'.

$$D_2(5, 8) = (8 + 2(5 - 8), 10 + 2(8 - 10)) = (8 + 2(-3), 10 + 2(-2)) = (8 - 6, 10 - 4) = (2, 6)$$

UNDERSTAND When the size of a figure changes in only one dimension, the transformation is called a stretch or a shrink. A vertical stretch pulls the points of a figure away from a horizontal line, such as the x-axis, and a vertical shrink pushes the points of the figure toward a horizontal line. A horizontal stretch pulls the points of a figure away from a vertical line, such as the y-axis, and a horizontal shrink pushes the points of the figure toward a vertical line. The lengths of line segments and the measure of angles usually change during a stretch or a shrink.



Connect

Dilate the line $y = 2x - 2$ using the rule $D_3(x, y) = (1 + 3(x - 1), 2 + 3(y - 2))$. Identify the scale factor and the center of dilation.

1 Dilate one point from the preimage.

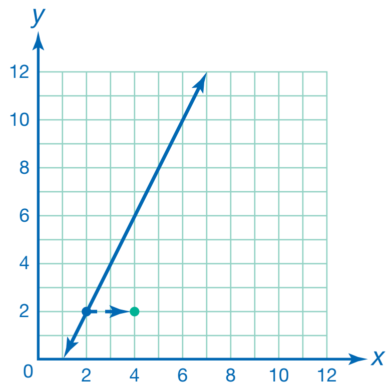
The point (2, 2) is on the preimage. In the function, let $x = 2$ and $y = 2$.

$D_3(2, 2) = (1 + 3(2 - 1), 2 + 3(2 - 2))$

$D_3(2, 2) = (1 + 3(1), 2 + 3(0))$

$D_3(2, 2) = (4, 2)$

Plot (4, 2) as the image of point (2, 2).



2 Dilate a second point and graph the image.

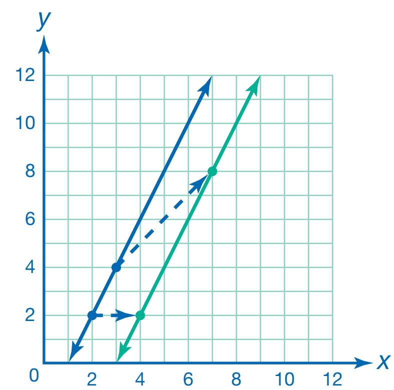
The point (3, 4) is also on the preimage.

$D_3(3, 4) = (1 + 3(3 - 1), 2 + 3(4 - 2))$

$D_3(3, 4) = (7, 8)$

Plot (7, 8) as the dilated image of (3, 4).

Two points is enough to determine a line, so draw a line through the two dilated points to graph the image.



3 Identify the scale factor and the center of dilation.

Compare the general rule for dilations with scale factor k and center of dilation (a, b) to the given rule.

General rule:
$D_k(x, y) = (a + k(x - a), b + k(y - b))$

Given rule:
$D_k(x, y) = (1 + 3(x - 1), 2 + 3(y - 2))$

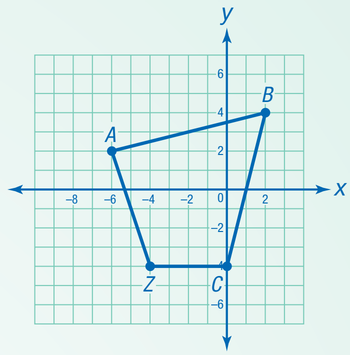
By comparing the formulas, we see that $k = 3$, $a = 1$, and $b = 2$.

▶ The scale factor is 3, and the center of dilation is (1, 2).

TRY

Dilate the same line given the function $D_k(x, y) = (1 + 3(x - 1), 3y)$. Identify the scale factor and center of dilation, and compare the image and preimage.

EXAMPLE A Quadrilateral *ABCZ* is shown. Draw quadrilateral $A'B'C'Z'$, which is the result of a dilation centered at the origin by a scale factor of $\frac{3}{2}$. Then write a function to describe the dilation.



1 Identify the coordinates of the vertices of quadrilateral *ABCZ*.

The quadrilateral has vertices $A(-6, 2)$, $B(2, 4)$, $C(0, -4)$, and $Z(-4, -4)$.

2 Find the coordinates of the vertices of the image.

This is a dilation from the origin, so multiply both the *x*- and *y*-coordinates by the scale factor, $\frac{3}{2}$.

$A(-6, 2) \rightarrow \left(\frac{3}{2}(-6), \frac{3}{2}(2)\right) \rightarrow A'(-9, 3)$

$B(2, 4) \rightarrow \left(\frac{3}{2}(2), \frac{3}{2}(4)\right) \rightarrow B'(3, 6)$

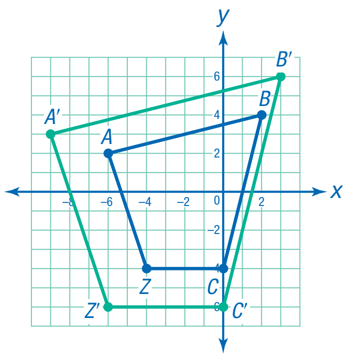
$C(0, -4) \rightarrow \left(\frac{3}{2}(0), \frac{3}{2}(-4)\right) \rightarrow C'(0, -6)$

$Z(-4, -4) \rightarrow \left(\frac{3}{2}(-4), \frac{3}{2}(-4)\right) \rightarrow Z'(-6, -6)$

3 Graph the image.

Plot the vertices and connect them.

▶



4 Use function notation to describe the dilation.

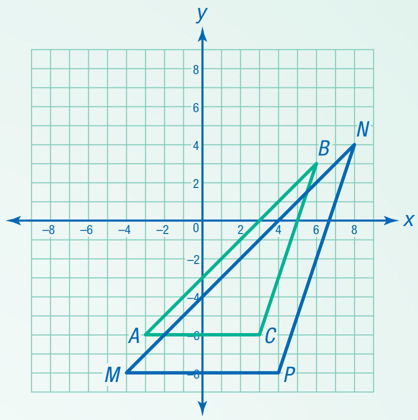
The function notation must show that each *x*-coordinate and each *y*-coordinate are multiplied by $\frac{3}{2}$.

▶ The function $D_{\frac{3}{2}}(x, y) = \left(\frac{3}{2}x, \frac{3}{2}y\right)$ represents the transformation.

DISCUSS

Compare the lengths of $\overline{ZC}$ and $\overline{Z'C'}$. How do you think the lengths of the other corresponding segments compare?

EXAMPLE B Triangles *MNP* and *ABC* are similar. Describe in words the dilation that would transform $\triangle MNP$ into $\triangle ABC$.



1 Determine the scale factor of the dilation.

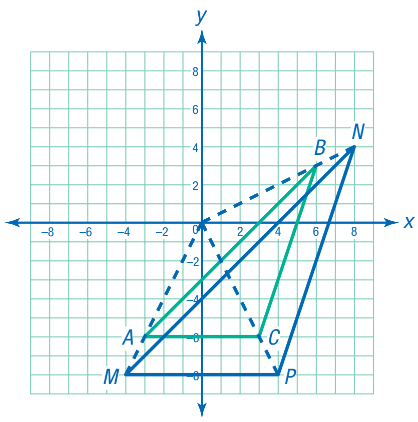
After dilation, corresponding sides of an image and its preimage are parallel. So, $\overline{AB}$ corresponds to $\overline{MN}$, $\overline{BC}$ corresponds to $\overline{NP}$, and $\overline{AC}$ corresponds to $\overline{MP}$.

Since $\overline{AC}$ and $\overline{MP}$ are horizontal, they can be measured simply by counting units. $AC = 6$ units and $MP = 8$ units. The scale factor of the dilation is equal to the ratio of the corresponding sides: $\frac{AC}{MP} = \frac{6}{8} = \frac{3}{4}$.

2 Locate the center of dilation.

The triangles do not share any vertices in common, so the center of dilation cannot be one of the vertices.

In order to locate the center of dilation, draw lines connecting corresponding vertices. The point at which they intersect is the center of dilation.



The lines appear to intersect at the origin.

3 Check your solution.

If the origin is the center of dilation, then each point (x, y) on $\triangle MNP$ will be dilated to form a corresponding point $\left(\frac{3}{4}x, \frac{3}{4}y\right)$ on $\triangle ABC$.

$M(-4, -8) \longrightarrow \left(\frac{3}{4}(-4), \frac{3}{4}(-8)\right) \rightarrow (-3, -6)$

$N(8, 4) \longrightarrow \left(\frac{3}{4}(8), \frac{3}{4}(4)\right) \rightarrow (6, 3)$

$P(4, -8) \longrightarrow \left(\frac{3}{4}(4), \frac{3}{4}(-8)\right) \rightarrow (3, -6)$

These coordinates are the vertices of $\triangle ABC$.

▶ A dilation of $\triangle MNP$ from the origin by a scale factor of $\frac{3}{4}$ would produce $\triangle ABC$.

TRY

Describe how the smaller triangle, $\triangle ABC$, could be dilated to form the larger triangle, $\triangle MNP$.

Practice

Write *true* or *false* for each statement. If false, rewrite the statement to make it true.

1. The dilation of a line segment by a scale factor greater than 1 results in a shorter segment.

2. Similar figures have corresponding sides that are congruent.

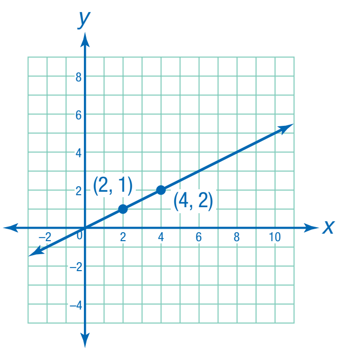
3. All regular pentagons are similar to one another.

4. When a line contains the center of dilation, its image is parallel to the preimage.

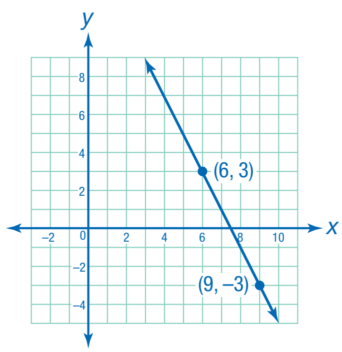
5. A vertical stretch of a figure will preserve the measure of the angles in the figure but not the lengths of its sides.

Graph the image that results from the dilation indicated. State if the resulting image is identical to the preimage, parallel to the preimage, or neither.

6. $D_2(x, y) = (2 + 2(x - 2), 1 + 2(y - 1))$



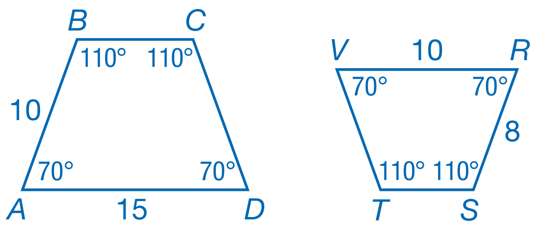
7. Scale factor: $\frac{1}{3}$
Center of dilation: the origin



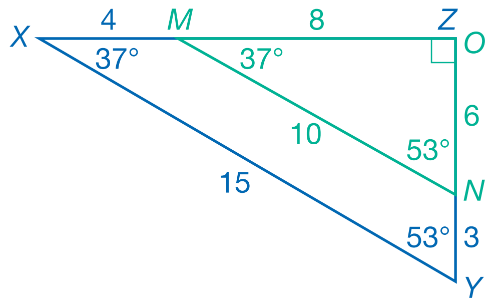
HINT Dilate two points on the line.

For questions 8 and 9, determine if the two figures are similar. Explain how you know.

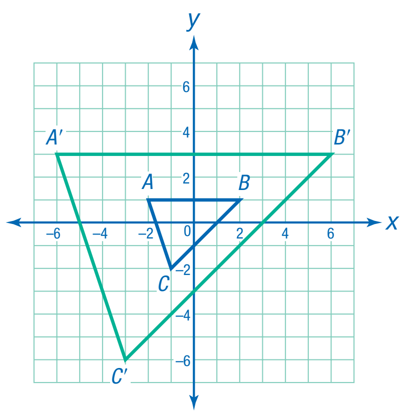
8. Trapezoids *ABCD* and *RSTV*



9. Triangles *XYZ* and *MNO*.



10. **APPLY** Describe how $\triangle ABC$ was transformed to its image, $\triangle A'B'C'$, both in words and in function notation.



Words:

Function Notation:

11. **IDENTIFY** Derek says that $\triangle GHJ$ could be transformed to $\triangle G'H'J'$ by dilating it from the origin by a scale factor of $\frac{2}{3}$. Is Derek correct? If not, identify the error Derek made and identify the correct dilation.